

C*-bialgebra defined as the direct sum of UHF algebras

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Abstract

Let $\mathcal{A}_0(*)$ denote the direct sum of a certain set of UHF algebras and let $\mathcal{A}(*) \equiv \mathbf{C} \oplus \mathcal{A}_0(*)$. We introduce a non-cocommutative comultiplication Δ_φ on $\mathcal{A}(*)$, and give an example of comodule-C*-algebra of the C*-bialgebra $(\mathcal{A}(*), \Delta_\varphi)$. With respect to Δ_φ , we define a non-symmetric tensor product of *-representations of UHF algebras and show tensor product formulas of GNS representations by product states.

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1 Introduction

A C*-bialgebra is a generalization of bialgebra in the theory of C*-algebras, which was introduced in C*-algebraic framework for quantum groups [26, 27]. We have studied C*-bialgebras and their construction method, and computed non-symmetric tensor products of *-representations with respect to non-cocommutative comultiplications [17, 18, 20, 21, 24, 22, 25]. In this paper, we introduce a non-cocommutative C*-bialgebra defined as the direct sum of a certain set of uniformly hyperfinite (=UHF) algebras. With respect to the comultiplication, we define a non-symmetric tensor product of *-representations of UHF algebras, and show tensor product formulas of Gel'fand-Naïmark-Segal (=GNS) representations by product states. The

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part of tensor product formulas has been given in the previous paper [23] without C^* -bialgebra. The present version is reorganized such that the tensor product is given by the comultiplication of a C^* -bialgebra. In this section, we show our motivation, definitions and construct the C^* -bialgebra. The main feature of this paper is as follows:

- A new non-commutative and non-cocommutative C^* -bialgebra is obtained. In our previous research, we treated only C^* -bialgebras (A, Δ) which satisfy $\Delta(A) \subset A \otimes A$. In this paper, this property does not holds. The bialgebra structure does not appear unless one takes the direct sum of all UHF algebras. Until now, there is no theory which treat all UHF algebras at once.
- The C^* -bialgebra is naturally constructed by using a well-known structure of UHF algebras. The standard parametrization of GNS representations of UHF algebras by product states is compatible with the tensor product formulas.
- Tensor product formulas of non-type I representations are obtained for the first time except [22].
- A construction method of C^* -bialgebra is a little bit generalized.

1.1 Motivation

In this subsection, we explain our motivation and the background of this study. Explicit mathematical definitions will be shown after § 1.2.

According to [12, 14, 30], given two representations of a group G , their tensor product (or Kronecker product [30]) is a new representation of G , which decomposes into a direct sum of indecomposable representations. The problem of finding this decomposition is called the *Clebsch-Gordan problem* and the resulting formula for the decomposition is called the *tensor product formula* (or *Clebsch-Gordan formula* [14]). Furthermore, the tensor product is important to describe the duality of G [33]. A generalization of the Clebsch-Gordan problem for groups is to consider modules over associative algebras instead of group algebras. However, there lies an obvious obstruction in that there is no known way to define the tensor product of two left modules over an arbitrary associative algebra. For group algebras, the extra structure coming from the group yields the tensor product. For a bialgebra A , the associative tensor product of representations of A can be defined by using the comultiplication. In this way, one of most important motivations of the study of bialgebras is the tensor product of their representations.

In [17], we introduced a non-symmetric tensor product among all $*$ -representations of Cuntz algebras and determined tensor product formulas of all permutative representations completely, in spite of the unknown of any comultiplication of Cuntz algebras. In [18], we generalized this construction of tensor product to a system of C^* -algebras and $*$ -homomorphisms indexed by a monoid. For example, we constructed a non-symmetric tensor product of all $*$ -representations of Cuntz-Krieger algebras by using Kronecker products of matrices [19, 20, 22].

On the other hand, UHF algebras and their $*$ -representations are well studied [2, 3, 4, 5, 8, 13, 29, 28]. For example, GNS representations of product states of UHF algebras were completely classified by [5]. This class contains $*$ -representations of UHF algebras of all Murray-von Neumann's types I, II, III ([6], § III.5).

Our interests are to construct a C^* -bialgebra from UHF algebras, and to define a tensor product of $*$ -representations of UHF algebras with respect to the comultiplication. Since one knows neither cocommutative nor non-cocommutative comultiplication of UHF algebras, the tensor product is new if one can find it.

1.2 C^* -bialgebra

In this subsection, we recall terminology about C^* -bialgebra according to [11, 26, 27]. For two C^* -algebras A and B , we write $\text{Hom}(A, B)$ as the set of all $*$ -homomorphisms from A to B , and let $\mathcal{M}(A)$ denote the multiplier algebra of A . We assume that every tensor product \otimes as below means the minimal C^* -tensor product.

Definition 1.1 *A pair (A, Δ) is a C^* -bialgebra if A is a C^* -algebra and $\Delta \in \text{Hom}(A, \mathcal{M}(A \otimes A))$ such that the linear span of $\{\Delta(a)(b \otimes c) : a, b, c \in A\}$ is norm dense in $A \otimes A$ and $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$. We call Δ the comultiplication of A .*

We say that a C^* -bialgebra (A, Δ) is *unital* if A is unital and Δ is unital; (A, Δ) is *counital* if there exists $\varepsilon \in \text{Hom}(A, \mathbf{C})$ which satisfies $(\varepsilon \otimes id) \circ \Delta = id = (id \otimes \varepsilon) \circ \Delta$. We call ε the *counit* of A and write (A, Δ, ε) as the counital C^* -bialgebra (A, Δ) with the counit ε . Remark that Definition 1.1 does not mean $\Delta(A) \subset A \otimes A$. A *bialgebra* in the purely algebraic theory [1, 16] means a unital counital bialgebra with the unital counit with respect to the algebraic tensor product, which does not need to have an involution. Hence a C^* -bialgebra is not a bialgebra in general.

According to [18], we recall several notions of C^* -bialgebra. A $*$ -homomorphism f from A to $\mathcal{M}(B)$ is *nondegenerate* if $f(A)B$ is dense in B . A pair (B, Γ) is a *right comodule- C^* -algebra* of a C^* -bialgebra (A, Δ) if B is a C^* -algebra and Γ is a nondegenerate $*$ -homomorphism from B to $\mathcal{M}(B \otimes A)$ which satisfies $(\Gamma \otimes id) \circ \Gamma = (id \otimes \Delta) \circ \Gamma$ where both $\Gamma \otimes id$ and $id \otimes \Delta$ are extended to unital $*$ -homomorphisms from $\mathcal{M}(B \otimes A)$ to $\mathcal{M}(B \otimes A \otimes A)$. In this case, the map Γ is called the *right coaction* of A on B . A C^* -bialgebra (A, Δ) is *proper* if $\Delta(a)(I \otimes b), (b \otimes I)\Delta(a) \in A \otimes A$ for any $a, b \in A$ where I denotes the unit of $\mathcal{M}(A)$. A proper C^* -bialgebra (A, Δ) satisfies the *cancellation law* if $\Delta(A)(I \otimes A)$ and $\Delta(A)(A \otimes I)$ are dense in $A \otimes A$ where $\Delta(A)(I \otimes A)$ and $\Delta(A)(A \otimes I)$ denote the linear spans of sets $\{\Delta(a)(I \otimes b) : a, b \in A\}$ and $\{\Delta(a)(b \otimes I) : a, b \in A\}$, respectively.

1.3 UHF algebras and $*$ -isomorphisms among their tensor products

In this subsection, we recall UHF algebras [13] and introduce a set of $*$ -isomorphisms among UHF algebras and their tensor products.

Let $\mathbf{N} \equiv \{1, 2, 3, \dots\}$, $\mathbf{N}_{\geq 2} \equiv \{2, 3, 4, \dots\}$ and let $\mathbf{N}_{\geq 2}^\infty$ denote the set of all sequences in $\mathbf{N}_{\geq 2}$. For $n \in \mathbf{N}$, let M_n denote the (finite-dimensional) C^* -algebra of all $n \times n$ -complex matrices. For $\mathbf{a} = (a_1, a_2, \dots) \in \mathbf{N}_{\geq 2}^\infty$, the sequence $\{M_{a_n}\}_{n \geq 1}$ of C^* -algebras defines the tensor product $\mathcal{A}_n(\mathbf{a}) \equiv \bigotimes_{j=1}^n M_{a_j}$. With respect to the embedding

$$\psi_{\mathbf{a}}^{(n)} : \mathcal{A}_n(\mathbf{a}) \ni A \mapsto A \otimes I \in \mathcal{A}_n(\mathbf{a}) \otimes M_{a_{n+1}} = \mathcal{A}_{n+1}(\mathbf{a}), \quad (1.1)$$

we regard $\mathcal{A}_n(\mathbf{a})$ as a C^* -subalgebra of $\mathcal{A}_{n+1}(\mathbf{a})$ and let $\mathcal{A}(\mathbf{a})$ denote the inductive limit of the system $\{(\mathcal{A}_n(\mathbf{a}), \psi_{\mathbf{a}}^{(n)}) : n \geq 1\}$:

$$\mathcal{A}(\mathbf{a}) \equiv \varinjlim (\mathcal{A}_n(\mathbf{a}), \psi_{\mathbf{a}}^{(n)}). \quad (1.2)$$

By definition, $\mathcal{A}(\mathbf{a})$ is a UHF algebra of Glimm's type $\{a_1, a_1 a_2, a_1 a_2 a_3, \dots\}$ which was classified by [13]. On the contrary, any UHF algebra is isomorphic to $\mathcal{A}(\mathbf{a})$ for some $\mathbf{a} \in \mathbf{N}_{\geq 2}^\infty$. Hence we call $\mathcal{A}(\mathbf{a})$ a UHF algebra in this paper.

Let $\{E_{i,j}^{(n)} : i, j = 1, \dots, n\}$ denote the set of standard matrix units of M_n . For $\mathbf{a} = (a_n), \mathbf{b} = (b_n) \in \mathbf{N}_{\geq 2}^\infty$, let $\mathbf{a} \cdot \mathbf{b} \equiv (a_1 b_1, a_2 b_2, \dots) \in \mathbf{N}_{\geq 2}^\infty$. Then $(\mathbf{N}_{\geq 2}^\infty, \cdot)$ is a commutative semigroup. For $\mathbf{a}, \mathbf{b} \in \mathbf{N}_{\geq 2}^\infty$, define the (standard) $*$ -isomorphism $\varphi_{\mathbf{a}, \mathbf{b}}^{(n)}$ from $\mathcal{A}_n(\mathbf{a} \cdot \mathbf{b})$ onto $\mathcal{A}_n(\mathbf{a}) \otimes \mathcal{A}_n(\mathbf{b})$ by

$$\varphi_{\mathbf{a}, \mathbf{b}}^{(n)}(E_{j_1, k_1}^{(a_1 b_1)} \otimes \dots \otimes E_{j_n, k_n}^{(a_n b_n)}) \equiv (E_{j'_1, k'_1}^{(a_1)} \otimes \dots \otimes E_{j'_n, k'_n}^{(a_n)}) \otimes (E_{j''_1, k''_1}^{(b_1)} \otimes \dots \otimes E_{j''_n, k''_n}^{(b_n)}) \quad (1.3)$$

for each $j_i, k_i \in \{1, \dots, a_i b_i\}$, $i = 1, \dots, n$ where $j'_1, \dots, j'_n, k'_1, \dots, k'_n, j''_1, \dots, j''_n, k''_1, \dots, k''_n$ are defined as $j_i = b_i(j'_i - 1) + j''_i$ and $k_i = b_i(k'_i - 1) + k''_i$, $j'_i, k'_i \in \{1, \dots, a_i\}$, $j''_i, k''_i \in \{1, \dots, b_i\}$ for each $i = 1, \dots, n$. For $\psi_{\mathbf{a}}^{(n)}$ in (1.1), we see that $(\psi_{\mathbf{a}}^{(n)} \otimes \psi_{\mathbf{b}}^{(n)}) \circ \varphi_{\mathbf{a}, \mathbf{b}}^{(n)} = \varphi_{\mathbf{a}, \mathbf{b}}^{(n+1)} \circ \psi_{\mathbf{a} \cdot \mathbf{b}}^{(n)}$ for each \mathbf{a}, \mathbf{b} and n . From this, we can define a unique $*$ -isomorphism $\varphi_{\mathbf{a}, \mathbf{b}}$ from $\mathcal{A}(\mathbf{a} \cdot \mathbf{b})$ onto $\mathcal{A}(\mathbf{a}) \otimes \mathcal{A}(\mathbf{b})$ such that

$$(\varphi_{\mathbf{a}, \mathbf{b}})|_{\mathcal{A}_n(\mathbf{a} \cdot \mathbf{b})} = \varphi_{\mathbf{a}, \mathbf{b}}^{(n)} \quad (n \geq 1) \quad (1.4)$$

where we identify $\mathcal{A}(\mathbf{a}) \otimes \mathcal{A}(\mathbf{b})$ with the inductive limit of the system $\{(\mathcal{A}_n(\mathbf{a}) \otimes \mathcal{A}_n(\mathbf{b}), \psi_{\mathbf{a}}^{(n)} \otimes \psi_{\mathbf{b}}^{(n)}) : n \geq 1\}$.

We add the unit $\mathbf{1} \equiv (1, 1, 1, \dots)$ for the semigroup $\mathbf{N}_{\geq 2}^{\infty}$ and write

$$\tilde{\mathbf{N}}_{\geq 2}^{\infty} \equiv \mathbf{N}_{\geq 2}^{\infty} \cup \{\mathbf{1}\}, \quad (1.5)$$

which is a subsemigroup of \mathbf{N}^{∞} . For convenience, define the 1-dimensional C^* -algebra

$$\mathcal{A}(\mathbf{1}) \equiv \mathbf{C}. \quad (1.6)$$

Remark that $\mathcal{A}(\mathbf{1})$ is not a UHF algebra by definition [13]. In addition, for $\mathbf{a} \in \mathbf{N}_{\geq 2}^{\infty}$, define $\varphi_{\mathbf{1}, \mathbf{1}}$, $\varphi_{\mathbf{1}, \mathbf{a}}$ and $\varphi_{\mathbf{a}, \mathbf{1}}$ by

$$\varphi_{\mathbf{1}, \mathbf{1}} = id_{\mathcal{A}(\mathbf{1})}, \quad \varphi_{\mathbf{1}, \mathbf{a}}(x) \equiv 1 \otimes x, \quad \varphi_{\mathbf{a}, \mathbf{1}}(x) \equiv x \otimes 1, \quad (x \in \mathcal{A}(\mathbf{a})) \quad (1.7)$$

where we identify $\mathcal{A}(\mathbf{1}) \otimes \mathcal{A}(\mathbf{1})$ with $\mathcal{A}(\mathbf{1})$.

Remark 1.2 We consider the meaning of (1.3). For two matrices $A \in M_n$ and $B \in M_m$, define the matrix $A \boxtimes B \in M_{nm}$ by

$$(A \boxtimes B)_{m(i-1)+i', m(j-1)+j'} \equiv A_{i,j} B_{i',j'} \quad (1.8)$$

for $i, j \in \{1, \dots, n\}$ and $i', j' \in \{1, \dots, m\}$. The new matrix $A \boxtimes B$ is called the *Kronecker product* of A and B [10, 32]. For $\mathbf{a}, \mathbf{b} \in \mathbf{N}_{\geq 2}^{\infty}$, we see that

$$\varphi_{\mathbf{a}, \mathbf{b}}^{(1)}(A \boxtimes B) = A \otimes B \quad (A \in M_{a_1}, B \in M_{b_1}). \quad (1.9)$$

Hence $\varphi_{\mathbf{a}, \mathbf{b}}^{(1)}$ is the inverse operation of the Kronecker product, which should be called the *Kronecker coproduct*.

1.4 Main theorems

In this subsection, we show our main theorems.

1.4.1 Construction of C*-bialgebra

In this subsection, we construct a C*-bialgebra. For the uncountable set $\{\mathcal{A}(\mathbf{a}) : \mathbf{a} \in \tilde{\mathbf{N}}_{\geq 2}^\infty\}$ of C*-algebras in (1.2) and (1.6), define the direct sum

$$\mathcal{A}(\ast) \equiv \bigoplus \{\mathcal{A}(\mathbf{a}) : \mathbf{a} \in \tilde{\mathbf{N}}_{\geq 2}^\infty\}. \quad (1.10)$$

For $\mathbf{a} \in \tilde{\mathbf{N}}_{\geq 2}^\infty$, define

$$\mathcal{N}_{\mathbf{a}} \equiv \{(\mathbf{b}, \mathbf{c}) \in \tilde{\mathbf{N}}_{\geq 2}^\infty \times \tilde{\mathbf{N}}_{\geq 2}^\infty : \mathbf{b} \cdot \mathbf{c} = \mathbf{a}\}, \quad (1.11)$$

and

$$\mathcal{A}^{(2)}(\mathbf{a}) \equiv \bigoplus \{\mathcal{A}(\mathbf{b}) \otimes \mathcal{A}(\mathbf{c}) : (\mathbf{b}, \mathbf{c}) \in \mathcal{N}_{\mathbf{a}}\}. \quad (1.12)$$

We see that $\mathcal{A}(\ast) \otimes \mathcal{A}(\ast) = \bigoplus \{\mathcal{A}^{(2)}(\mathbf{a}) : \mathbf{a} \in \tilde{\mathbf{N}}_{\geq 2}^\infty\}$. For $\{\varphi_{\mathbf{a}, \mathbf{b}} : \mathbf{a}, \mathbf{b} \in \tilde{\mathbf{N}}_{\geq 2}^\infty\}$ in (1.4), define the \ast -homomorphism $\Delta_\varphi^{(\mathbf{a})}$ from $\mathcal{A}(\mathbf{a})$ to $\mathcal{M}(\mathcal{A}^{(2)}(\mathbf{a}))$ by

$$\Delta_\varphi^{(\mathbf{a})}(x) \equiv \prod_{(\mathbf{b}, \mathbf{c}) \in \mathcal{N}_{\mathbf{a}}} \varphi_{\mathbf{b}, \mathbf{c}}(x) \quad (x \in \mathcal{A}(\mathbf{a})) \quad (1.13)$$

where we identify $\mathcal{M}(\mathcal{A}^{(2)}(\mathbf{a}))$ with the direct product $\prod_{(\mathbf{b}, \mathbf{c}) \in \mathcal{N}_{\mathbf{a}}} \mathcal{A}(\mathbf{b}) \otimes \mathcal{A}(\mathbf{c})$. Define the \ast -homomorphism Δ_φ from $\mathcal{A}(\ast)$ to $\mathcal{M}(\mathcal{A}(\ast) \otimes \mathcal{A}(\ast))$ by

$$\Delta_\varphi \equiv \bigoplus \{\Delta_\varphi^{(\mathbf{a})} : \mathbf{a} \in \tilde{\mathbf{N}}_{\geq 2}^\infty\} \quad (1.14)$$

where we also identify $\bigoplus \{\mathcal{M}(\mathcal{A}^{(2)}(\mathbf{a})) : \mathbf{a} \in \tilde{\mathbf{N}}_{\geq 2}^\infty\}$ with a C*-subalgebra of $\mathcal{M}(\mathcal{A}(\ast) \otimes \mathcal{A}(\ast)) \cong \prod \{\mathcal{M}(\mathcal{A}^{(2)}(\mathbf{a})) : \mathbf{a} \in \tilde{\mathbf{N}}_{\geq 2}^\infty\}$.

Theorem 1.3 *Let $(\mathcal{A}(\ast), \Delta_\varphi)$ be as in (1.10) and (1.14), and let ε denote the projection from $\mathcal{A}(\ast)$ onto $\mathcal{A}(\mathbf{1})$. Then the following holds:*

- (i) $(\mathcal{A}(\ast), \Delta_\varphi)$ is a non-cocommutative proper C*-bialgebra with counit ε .
- (ii) $(\mathcal{A}(\ast), \Delta_\varphi)$ satisfies the cancellation law.

Remark 1.4 (i) The idea of the definition of $\{\varphi_{\mathbf{a}, \mathbf{b}}\}$ in (1.4) is an analogy of the set of embeddings of Cuntz algebras in § 1.2 of [17]. In § 6.1 of [18], we also defined a C*-bialgebra defined as the direct sum of a countably infinite set of UHF algebras:

$$UHF_\ast \equiv \mathbf{C} \oplus UHF_2 \oplus UHF_3 \oplus UHF_4 \oplus \cdots \quad (1.15)$$

where UHF_n is defined as the fixed point subalgebra of the Cuntz algebra \mathcal{O}_n with respect to the $U(1)$ -gauge action, which is $*$ -isomorphic onto the inductive limit $\varinjlim_k M_n^{\otimes k}$. Clearly, UHF_* is a C^* -subalgebra of $\mathcal{A}(*)$ in (1.10), but *not* a C^* -subbialgebra. The reason is as follows: The comultiplication Δ of UHF_* in [18] satisfies $\Delta(UHF_*) \subset UHF_* \otimes UHF_*$. On the other hand, the restriction $\Delta_\varphi|_{UHF_*}$ of the comultiplication Δ_φ of $\mathcal{A}(*)$ in (1.14) satisfies $\Delta_\varphi(UHF_*) \not\subset UHF_* \otimes UHF_*$.

- (ii) In order to help reader's understanding, we demonstrate the image of comultiplication Δ_φ according to the definition. Let $\mathbf{6} = (6, 6, \dots) \in \tilde{\mathbf{N}}_{\geq 2}^\infty$. Then

$$\mathcal{N}_{\mathbf{6}} = \left\{ (\mathbf{6}, \mathbf{1}), (\mathbf{1}, \mathbf{6}), (\mathbf{a}, \bar{\mathbf{a}}) : \begin{array}{l} \mathbf{a} = (a_1, a_2, \dots), a_i \in \{2, 3\}, i \geq 1 \\ \bar{\mathbf{a}} = (6/a_1, 6/a_2, \dots) \end{array} \right\}. \quad (1.16)$$

Remark that $\mathcal{N}_{\mathbf{6}}$ is also a uncountable set. When $(\mathbf{b}, \mathbf{c}) \in \mathcal{N}_{\mathbf{6}}$ and $\mathbf{b} = (b_1, b_2, b_3, \dots)$, b_1 is 1 or 2 or 3 or 6. Recall $\mathcal{A}(\mathbf{6}) = \varinjlim \mathcal{A}_n(\mathbf{6})$ and $\mathcal{A}_1(\mathbf{6}) = M_6 \supset \{E_{i,j}^{(6)} : i, j = 1, \dots, 6\}$. For $x \in \mathcal{A}_1(\mathbf{6})$,

$$\Delta_\varphi(x) = \Delta_\varphi^{(6)}(x) = \prod_{(\mathbf{b}, \mathbf{c}) \in \mathcal{N}_{\mathbf{6}}} \varphi_{\mathbf{b}, \mathbf{c}}(x) = \prod_{(\mathbf{b}, \mathbf{c}) \in \mathcal{N}_{\mathbf{6}}} \varphi_{\mathbf{b}, \mathbf{c}}^{(1)}(x). \quad (1.17)$$

Since $\Delta_\varphi(x) \in \mathcal{M}(\mathcal{A}^{(2)}(\mathbf{a})) = \prod_{(\mathbf{b}, \mathbf{c}) \in \mathcal{N}_{\mathbf{6}}} \mathcal{A}(\mathbf{b}) \otimes \mathcal{A}(\mathbf{c})$, we write $\Delta_\varphi(x)$ as an element $(x_{\mathbf{b}, \mathbf{c}})_{(\mathbf{b}, \mathbf{c}) \in \mathcal{N}_{\mathbf{6}}}$ in $\prod_{(\mathbf{b}, \mathbf{c}) \in \mathcal{N}_{\mathbf{6}}} \mathcal{A}(\mathbf{b}) \otimes \mathcal{A}(\mathbf{c})$ (see § 2.1). Especially, we compute the case $x = E_{2,2}^{(6)}$. Since $E_{2,2}^{(6)} = E_{1,1}^{(2)} \boxtimes E_{2,2}^{(3)} = E_{1,1}^{(3)} \boxtimes E_{2,2}^{(2)}$ and (1.9), components of $\Delta_\varphi(x)$ are given as follows:

$$x_{\mathbf{b}, \mathbf{c}} = \begin{cases} 1 \otimes E_{2,2}^{(6)} & (\text{when } b_1 = 1), \\ E_{1,1}^{(2)} \otimes E_{2,2}^{(3)} & (\text{when } b_1 = 2), \\ E_{1,1}^{(3)} \otimes E_{2,2}^{(2)} & (\text{when } b_1 = 3), \\ E_{2,2}^{(6)} \otimes 1 & (\text{when } b_1 = 6). \end{cases} \quad (1.18)$$

This shows that the flip of the element $\Delta_\varphi(E_{2,2}^{(6)})$ in $\mathcal{M}(\mathcal{A}(*) \otimes \mathcal{A}(*))$ is not equal to $\Delta_\varphi(E_{2,2}^{(6)})$. Hence Δ_φ is non-cocommutative.

1.4.2 Construction of comodule-C*-algebra

Next, we introduce an example of comodule-C*-algebra of $(\mathcal{A}(*), \Delta_\varphi)$. Let \mathcal{O}_∞ denote the Cuntz algebra generated by the canonical generators $\{s_i : i \in \mathbf{N}\}$ [9]. For two sequences $J = (j_1, \dots, j_n), K = (k_1, \dots, k_n) \in \mathbf{N}^n$, define $E_{J,K}^{(\infty)} \equiv s_{j_1} \cdots s_{j_n} s_{k_n}^* \cdots s_{k_1}^* \in \mathcal{O}_\infty$ and define UHF_∞ as the unital C*-subalgebra of \mathcal{O}_∞ generated by $\bigcup_{n \geq 1} \{E_{J,K}^{(\infty)} : J, K \in \mathbf{N}^n\}$:

$$UHF_\infty \equiv C^* \langle \bigcup_{n \geq 1} \{E_{J,K}^{(\infty)} : J, K \in \mathbf{N}^n\} \rangle \subset \mathcal{O}_\infty. \quad (1.19)$$

We prepare some new notations. For $\mathbf{a} = (a_1, a_2, \dots) \in \tilde{\mathbf{N}}_{\geq 2}^\infty$ and $n \geq 1$, define the finite subset $S_n(\mathbf{a})$ of \mathbf{N}^n by

$$S_n(\mathbf{a}) \equiv \{1, \dots, a_1\} \times \cdots \times \{1, \dots, a_n\}. \quad (1.20)$$

Let $\{E_{j,k}^{(n)}\}$ be as in § 1.3. For $J = (j_i), K = (k_i) \in S_n(\mathbf{a})$, define $E_{J,K}^{(\mathbf{a})} \in \mathcal{A}_n(\mathbf{a})$ by

$$E_{J,K}^{(\mathbf{a})} \equiv E_{j_1, k_1}^{(a_1)} \otimes \cdots \otimes E_{j_n, k_n}^{(a_n)}. \quad (1.21)$$

Define the *-homomorphism $\varphi_{\infty, \mathbf{a}}$ from UHF_∞ to $UHF_\infty \otimes \mathcal{A}(\mathbf{a})$ by

$$\varphi_{\infty, \mathbf{a}}(E_{J,K}^{(\infty)}) \equiv E_{J', K'}^{(\infty)} \otimes E_{J'', K''}^{(\mathbf{a})} \quad (J, K \in \mathbf{N}^n, n \geq 1) \quad (1.22)$$

where $J' = (j'_i), K' = (k'_i) \in \mathbf{N}^n$ and $J'' = (j''_i), K'' = (k''_i) \in S_n(\mathbf{a})$ are defined as $j_i = a_i(j'_i - 1) + j''_i$ and $k_i = a_i(k'_i - 1) + k''_i$ for $i = 1, \dots, n$.

Theorem 1.5 *Let UHF_∞ and $\{\varphi_{\infty, \mathbf{a}} : \mathbf{a} \in \tilde{\mathbf{N}}_{\geq 2}^\infty\}$ be as in (1.19) and (1.22), respectively. Define the *-homomorphism Γ_φ from UHF_∞ to $\mathcal{M}(UHF_\infty \otimes \mathcal{A}(*))$ by*

$$\Gamma_\varphi(x) \equiv \prod_{\mathbf{a} \in \tilde{\mathbf{N}}_{\geq 2}^\infty} \varphi_{\infty, \mathbf{a}}(x) \quad (x \in UHF_\infty) \quad (1.23)$$

where we identify $\mathcal{M}(UHF_\infty \otimes \mathcal{A}())$ with $\prod \{UHF_\infty \otimes \mathcal{A}(\mathbf{a}) : \mathbf{a} \in \tilde{\mathbf{N}}_{\geq 2}^\infty\}$. Then $(UHF_\infty, \Gamma_\varphi)$ is a right comodule-C*-algebra of $(\mathcal{A}(*), \Delta_\varphi)$.*

In § 2, we will prove theorems in § 1.4. In § 3, we will show tensor product formulas among *-representations of UHF algebras, and show more concrete tensor product formulas for special UHF algebras.

2 Proofs of main theorems

In this section, we show a general method to construct C^* -bialgebras and prove theorems in § 1.4.

2.1 C^* -weakly coassociative system

In this subsection, we review a general method to construct C^* -bialgebras [18, 25], and generalize it in order to prove theorems.

First, we recall basic facts of the direct product and the direct sum of general C^* -algebras [6]. We define two C^* -algebras $\prod_{i \in \Omega} A_i$ and $\bigoplus_{i \in \Omega} A_i$ as follows: $\prod_{i \in \Omega} A_i \equiv \{(a_i) : \|(a_i)\| \equiv \sup_i \|a_i\| < \infty\}$, $\bigoplus_{i \in \Omega} A_i \equiv \{(a_i) : \|a_i\| \rightarrow 0 \text{ as } i \rightarrow \infty\}$. We call $\prod_{i \in \Omega} A_i$ and $\bigoplus_{i \in \Omega} A_i$ the *direct product* and the *direct sum* of A_i 's, respectively. It is known that $\mathcal{M}(\bigoplus_{i \in \Omega} A_i) \cong \prod_{i \in \Omega} \mathcal{M}(A_i)$. If A_i is unital for each i , then $\mathcal{M}(\bigoplus_{i \in \Omega} A_i) \cong \prod_{i \in \Omega} A_i$.

Let $\{B_i : i \in \Omega\}$ be another set of C^* -algebras and let $\{f_i : i \in \Omega\}$ be a set of $*$ -homomorphisms such that $f_i \in \text{Hom}(A_i, B_i)$ for each $i \in \Omega$. Then we obtain $\bigoplus_{i \in \Omega} f_i \in \text{Hom}(\bigoplus_{i \in \Omega} A_i, \bigoplus_{i \in \Omega} B_i)$. If f_i is nondegenerate for each i , then $\bigoplus_{i \in \Omega} f_i$ is also nondegenerate. If both A_i and B_i are unital and f_i is unital for each $i \in \Omega$, then $\bigoplus_{i \in \Omega} f_i$ is nondegenerate.

A *monoid* is a set M equipped with a binary associative operation $M \times M \ni (a, b) \mapsto ab \in M$, and a unit with respect to the operation. We recall the definition of C^* -weakly coassociative system in [25].

Definition 2.1 *Let M be a monoid with the unit e . A data $\{(A_a, \varphi_{a,b}) : a, b \in M\}$ is a C^* -weakly coassociative system (= C^* -WCS) over M if A_a is a unital C^* -algebra for $a \in M$ and $\varphi_{a,b}$ is a unital $*$ -homomorphism from A_{ab} to $A_a \otimes A_b$ for $a, b \in M$ such that*

- (i) *for all $a, b, c \in M$, the following holds:*

$$(id_a \otimes \varphi_{b,c}) \circ \varphi_{a,bc} = (\varphi_{a,b} \otimes id_c) \circ \varphi_{ab,c} \quad (2.1)$$

where id_x denotes the identity map on A_x for $x = a, c$,

- (ii) *there exists a counit ε_e of A_e such that $(A_e, \varphi_{e,e}, \varepsilon_e)$ is a counital C^* -bialgebra,*

- (iii) *for each $a \in M$, the following holds:*

$$(\varepsilon_e \otimes id_a) \circ \varphi_{e,a} = id_a = (id_a \otimes \varepsilon_e) \circ \varphi_{a,e}. \quad (2.2)$$

We slightly generalize Theorem 2.2 in [25] as follows.

Theorem 2.2 Let $\{(A_a, \varphi_{a,b}) : a, b \in \mathbf{M}\}$ be a C^* -WCS over a monoid \mathbf{M} . Define C^* -algebras

$$A_* \equiv \oplus \{A_a : a \in \mathbf{M}\}, \quad C_a \equiv \oplus \{A_b \otimes A_c : (b, c) \in \mathcal{N}_a\} \quad (a \in \mathbf{M}) \quad (2.3)$$

where $\mathcal{N}_a \equiv \{(b, c) \in \mathbf{M} \times \mathbf{M} : bc = a\}$. Define $\Delta_\varphi^{(a)} \in \text{Hom}(A_a, \mathcal{M}(C_a))$, $\Delta_\varphi \in \text{Hom}(A_*, \mathcal{M}(A_* \otimes A_*))$ and $\varepsilon \in \text{Hom}(A_*, \mathbf{C})$ by

$$\left\{ \begin{array}{l} \Delta_\varphi^{(a)}(x) \equiv \prod_{(b,c) \in \mathcal{N}_a} \varphi_{b,c}(x) \quad (x \in A_a), \\ \Delta_\varphi \equiv \oplus \{\Delta_\varphi^{(a)} : a \in \mathbf{M}\}, \\ \varepsilon \equiv \varepsilon_e \circ E_e \end{array} \right. \quad (2.4)$$

where we identify $\mathcal{M}(A_* \otimes A_*)$ with $\prod \{\mathcal{M}(C_a) : a \in \mathbf{M}\}$, and E_e denotes the projection from A_* onto A_e . Then the following holds:

- (i) $(A_*, \Delta_\varphi, \varepsilon)$ is a proper counital C^* -bialgebra.
- (ii) In addition, if $\#\mathcal{N}_a < \infty$ for each $a \in \mathbf{M}$, then $\Delta_\varphi(A_*) \subset A_* \otimes A_*$ where we naturally identify $A_* \otimes A_*$ with a C^* -subalgebra of $\mathcal{M}(A_* \otimes A_*)$.

Proof. (i) From (2.3), $\mathcal{M}(C_a) = \prod \{A_b \otimes A_c : (b, c) \in \mathcal{N}_a\}$. Hence $\Delta_\varphi^{(a)}$ is well-defined. Since $\mathcal{M}(A_* \otimes A_*) = \prod_{a,b \in \mathbf{M}} A_a \otimes A_b$, Δ_φ is also well-defined. We show the coassociativity of Δ_φ . Let $a, b, c \in \mathbf{M}$ and let $Y \in A_a \otimes A_b \otimes A_c$ and $x \in A_{abc}$. From (2.1) and (2.4), we can verify that

$$\{(\Delta_\varphi \otimes id) \circ \Delta_\varphi\}(x)Y = \{(id \otimes \Delta_\varphi) \circ \Delta_\varphi\}(x)Y. \quad (2.5)$$

This implies $\{(\Delta_\varphi \otimes id) \circ \Delta_\varphi\}(x) = \{(id \otimes \Delta_\varphi) \circ \Delta_\varphi\}(x)$ on $A_* \otimes A_* \otimes A_*$ for each $x \in A_*$. Hence the coassociativity is verified. As the same token, we can verify that ε is a counit and (A_*, Δ_φ) is proper.

(ii) This follows from Theorem 2.2 of [25]. ■

We call $(A_*, \Delta_\varphi, \varepsilon)$ in Theorem 2.2 by a (counital) C^* -bialgebra associated with $\{(A_a, \varphi_{a,b}) : a, b \in \mathbf{M}\}$.

We prepare lemmas as follows.

Lemma 2.3 (i) Let $\{(A_a, \varphi_{a,b}) : a, b \in \mathbf{M}\}$ be a C^* -WCS over a monoid \mathbf{M} and let (A_*, Δ_φ) be as in Theorem 2.2 associated with $\{(A_a, \varphi_{a,b}) :$

$a, b \in \mathbf{M}$. For $a, b \in \mathbf{M}$, let I_a denote the unit of A_a for $a \in \mathbf{M}$, and define

$$X_{a,b} \equiv \varphi_{a,b}(A_{ab})(A_a \otimes I_b), \quad Y_{a,b} \equiv \varphi_{a,b}(A_{ab})(I_a \otimes A_b) \quad (2.6)$$

where $\varphi_{a,b}(A_{ab})(A_a \otimes I_b)$ and $\varphi_{a,b}(A_{ab})(I_a \otimes A_b)$ mean the linear spans of $\{\varphi_{a,b}(x)(y \otimes I_b) : x \in A_{ab}, y \in A_a\}$ and $\{\varphi_{a,b}(x)(I_a \otimes y) : x \in A_{ab}, y \in A_b\}$, respectively. If both $X_{a,b}$ and $Y_{a,b}$ are dense in $A_a \otimes A_b$ for each $a, b \in \mathbf{M}$, then (A_*, Δ_φ) satisfies the cancellation law.

- (ii) For a C^* -WCS $\{(A_a, \varphi_{a,b}) : a, b \in \mathbf{M}\}$ over a monoid \mathbf{M} , assume that B is a unital C^* -algebra and a set $\{\varphi_{B,a} : a \in \mathbf{M}\}$ of unital $*$ -homomorphisms such that $\varphi_{B,a} \in \text{Hom}(B, B \otimes A_a)$ for each $a \in \mathbf{M}$ and the following holds:

$$(\varphi_{B,a} \otimes id_b) \circ \varphi_{B,b} = (id_B \otimes \varphi_{a,b}) \circ \varphi_{B,ab} \quad (a, b \in \mathbf{M}). \quad (2.7)$$

Then B is a right comodule- C^* -algebra of the C^* -bialgebra (A_*, Δ_φ) with the unital coaction $\Gamma_\varphi \equiv \prod_{a \in \mathbf{M}} \varphi_{B,a}$.

Proof. (i) By definition, the algebraic direct sum $Q \equiv \oplus_{alg} \{\Delta_\varphi(A_c)(A_a \otimes I) : c, a \in \mathbf{M}\}$ is dense in $\Delta_\varphi(A_*)(A_* \otimes I)$. On the other hand,

$$\Delta_\varphi(A_c)(A_a \otimes I) = \begin{cases} \varphi_{a,b}(A_{ab})(A_a \otimes I_b) & (\exists b \in \mathbf{M} \text{ s.t. } ab = c), \\ 0 & (\text{otherwise}). \end{cases} \quad (2.8)$$

From this,

$$Q = \oplus_{alg} \{\varphi_{a,b}(A_{ab})(A_a \otimes I_b) : a, b \in \mathbf{M}\} = \oplus_{alg} \{X_{a,b} : a, b \in \mathbf{M}\}. \quad (2.9)$$

By assumption, $\oplus_{alg} \{X_{a,b} : a, b \in \mathbf{M}\}$ is dense in $\oplus_{alg} \{A_a \otimes A_b : a, b \in \mathbf{M}\}$. Hence $\Delta_\varphi(A_*)(A_* \otimes I)$ is dense in $A_* \otimes A_*$. In a similar fashion, we see that $\Delta_\varphi(A_*)(I \otimes A_*)$ is also dense in $A_* \otimes A_*$. Hence the statement holds.

(ii) By identifying $\mathcal{M}(B \otimes A_*)$ with $\prod \{B \otimes A_a : a \in \mathbf{M}\}$, Γ_φ is well-defined. From (2.7), we can verify that $(\Gamma_\varphi \otimes id_{A_*}) \circ \Gamma_\varphi = (id_B \otimes \Delta_\varphi) \circ \Gamma_\varphi$. Hence the statement holds. \blacksquare

2.2 Proofs of theorems

In this subsection, we prove theorems in § 1.4.

Proof of Theorem 1.3. (i) From Theorem 2.2(i), it is sufficient to show that $\{(\mathcal{A}(\mathbf{a}), \varphi_{\mathbf{a}, \mathbf{b}}) : \mathbf{a}, \mathbf{b} \in \tilde{\mathbf{N}}_{\geq 2}^\infty\}$ in (1.2) and (1.4) is a \mathbf{C}^* -WCS over the monoid $\tilde{\mathbf{N}}_{\geq 2}^\infty$. By definition, the following holds:

$$(\varphi_{\mathbf{a}, \mathbf{b}} \otimes id_{\mathbf{c}}) \circ \varphi_{\mathbf{a}, \mathbf{b}, \mathbf{c}} = (id_{\mathbf{a}} \otimes \varphi_{\mathbf{b}, \mathbf{c}}) \circ \varphi_{\mathbf{a}, \mathbf{b}, \mathbf{c}} \quad (\mathbf{a}, \mathbf{b}, \mathbf{c} \in \tilde{\mathbf{N}}_{\geq 2}^\infty) \quad (2.10)$$

where $id_{\mathbf{x}}$ denotes the identity map on $\mathcal{A}(\mathbf{x})$ for $\mathbf{x} = \mathbf{a}, \mathbf{c}$. Equivalently, the following diagram is commutative:

Figure 2.4

$$\begin{array}{ccccc}
 & & \mathcal{A}(\mathbf{a}) \otimes \mathcal{A}(\mathbf{b} \cdot \mathbf{c}) & & \\
 & \nearrow \varphi_{\mathbf{a}, \mathbf{b}, \mathbf{c}} & & \searrow id_{\mathbf{a}} \otimes \varphi_{\mathbf{b}, \mathbf{c}} & \\
 \mathcal{A}(\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}) & & & & \mathcal{A}(\mathbf{a}) \otimes \mathcal{A}(\mathbf{b}) \otimes \mathcal{A}(\mathbf{c}). \\
 & \searrow \varphi_{\mathbf{a}, \mathbf{b}, \mathbf{c}} & & \nearrow \varphi_{\mathbf{a}, \mathbf{b}} \otimes id_{\mathbf{c}} & \\
 & & \mathcal{A}(\mathbf{a} \cdot \mathbf{b}) \otimes \mathcal{A}(\mathbf{c}) & &
 \end{array}$$

Especially, $(\mathcal{A}(\mathbf{1}), \varphi_{\mathbf{1}, \mathbf{1}})$ is a one-dimensional \mathbf{C}^* -bialgebra with counit $id_{\mathcal{A}(\mathbf{1})}$. Hence $\{(\mathcal{A}(\mathbf{a}), \varphi_{\mathbf{a}, \mathbf{b}}) : \mathbf{a}, \mathbf{b} \in \tilde{\mathbf{N}}_{\geq 2}^\infty\}$ is a \mathbf{C}^* -WCS. The non-cocommutativity of $(\mathcal{A}(*), \Delta_\varphi)$ has been shown in Remark 1.4(ii).

(ii) For $\mathbf{a}, \mathbf{b} \in \tilde{\mathbf{N}}_{\geq 2}^\infty$, let $X_{\mathbf{a}, \mathbf{b}} \equiv \varphi_{\mathbf{a}, \mathbf{b}}(\mathcal{A}(\mathbf{a} \cdot \mathbf{b}))(\mathcal{A}(\mathbf{a}) \otimes I_{\mathbf{b}})$ and $Y_{\mathbf{a}, \mathbf{b}} \equiv \varphi_{\mathbf{a}, \mathbf{b}}(\mathcal{A}(\mathbf{a} \cdot \mathbf{b}))(I_{\mathbf{a}} \otimes \mathcal{A}(\mathbf{b}))$. From Lemma 2.3(i), it is sufficient to show that both $X_{\mathbf{a}, \mathbf{b}}$ and $Y_{\mathbf{a}, \mathbf{b}}$ are dense in $\mathcal{A}(\mathbf{a}) \otimes \mathcal{A}(\mathbf{b})$.

By definition, $X_{\mathbf{a}, \mathbf{b}}$ is the linear span of $\{\varphi_{\mathbf{a}, \mathbf{b}}(x)(y \otimes I_{\mathbf{b}}) : x \in \mathcal{A}(\mathbf{a} \cdot \mathbf{b}), y \in \mathcal{A}(\mathbf{a})\}$. Let $S_n(\mathbf{a})$ and $E_{J, K}^{(\mathbf{a})}$ be as in (1.20) and (1.21), respectively. We see that $\mathcal{A}(\mathbf{a}) \otimes \mathcal{A}(\mathbf{b})$ is linearly spanned by the set

$$\bigcup_{n, m \geq 1} \{E_{J', K'}^{(\mathbf{a})} \otimes E_{J'', K''}^{(\mathbf{b})} : J', K' \in S_n(\mathbf{a}), J'', K'' \in S_m(\mathbf{b})\} \quad (2.11)$$

as a Banach space.

For $n, m \geq 1$, fix $J', K' \in S_n(\mathbf{a})$ and $J'', K'' \in S_m(\mathbf{b})$.

- (a) If $n = m$, then we can choose J and K in $S_n(\mathbf{a} \cdot \mathbf{b})$ such that $E_{J, K}^{(\mathbf{a} \cdot \mathbf{b})} = E_{J', K'}^{(\mathbf{a})} \boxtimes E_{J'', K''}^{(\mathbf{b})}$ where \boxtimes means the componentwise Kronecker product. Since $E_{J', K'}^{(\mathbf{a})} \otimes E_{J'', K''}^{(\mathbf{b})} = \Delta_\varphi(E_{J, K}^{(\mathbf{a} \cdot \mathbf{b})})(E_{J', K'}^{(\mathbf{a})} \otimes I_{\mathbf{b}})$, we see $E_{J', K'}^{(\mathbf{a})} \otimes E_{J'', K''}^{(\mathbf{b})} \in \Delta_\varphi(\mathcal{A}(\mathbf{a} \cdot \mathbf{b}))(\mathcal{A}(\mathbf{a}) \otimes I_{\mathbf{b}})$.

(b) If $n - m = k > 0$, then we can write as

$$E_{J',K'}^{(\mathbf{a})} \otimes E_{J'',K''}^{(\mathbf{b})} = \sum_{L \in S_{m,k}(\mathbf{b})} E_{J',K'}^{(\mathbf{a})} \otimes E_{(J''L),(K''L)}^{(\mathbf{b})} \quad (2.12)$$

where $S_{m,k}(\mathbf{b}) \equiv \{1, \dots, b_{m+1}\} \times \dots \times \{1, \dots, b_{m+k}\}$ for $\mathbf{b} = (b_1, b_2, \dots)$ and $(J''L)$ denotes the concatenation of two sequences J'' and L . The right hand side of (2.12) is contained in $\Delta_\varphi(\mathcal{A}(\mathbf{a} \cdot \mathbf{b}))(\mathcal{A}(\mathbf{a}) \otimes I_{\mathbf{b}})$ from (a).

(c) If $n - m = -k < 0$, then this follows from (b) by the same token.

From (a),(b),(c), $\Delta_\varphi(\mathcal{A}(\mathbf{a} \cdot \mathbf{b}))(\mathcal{A}(\mathbf{a}) \otimes I_{\mathbf{b}})$ is dense in $\mathcal{A}(\mathbf{a}) \otimes \mathcal{A}(\mathbf{b})$. Just the same, we see that $\Delta_\varphi(\mathcal{A}(\mathbf{a} \cdot \mathbf{b}))(I_{\mathbf{a}} \otimes \mathcal{A}(\mathbf{b}))$ is dense in $\mathcal{A}(\mathbf{a}) \otimes \mathcal{A}(\mathbf{b})$. Hence the statement holds. \blacksquare

Proof of Theorem 1.5. Let $\mathbf{a}, \mathbf{b} \in \tilde{\mathbf{N}}_{\geq 2}^\infty$, $Y \in UHF_\infty \otimes \mathcal{A}(\mathbf{a}) \otimes \mathcal{A}(\mathbf{b})$ and $x \in UHF_\infty$. By definition, we see that

$$\{(\varphi_{\infty, \mathbf{a}} \otimes id_{\mathbf{b}}) \circ \varphi_{\infty, \mathbf{b}}\}(x)Y = \{(id_\infty \otimes \varphi_{\mathbf{a}, \mathbf{b}}) \circ \varphi_{\infty, \mathbf{a} \cdot \mathbf{b}}\}(x)Y \quad (2.13)$$

where id_∞ denotes the identity map on UHF_∞ . From this and Lemma 2.3(ii) for $B = UHF_\infty$ and $\varphi_{B, \mathbf{a}} \equiv \varphi_{\infty, \mathbf{a}}$, the statement holds. \blacksquare

3 Tensor product formulas

In this section, we show tensor product formulas of $*$ -representations of the C^* -algebra $\mathcal{A}(*)$ with respect to the comultiplication Δ_φ in § 1.4.

3.1 Basic properties

In this subsection, we introduce a tensor product of $*$ -representations and that of states of UHF algebras, and show its basic properties. For a C^* -algebra \mathfrak{A} , let $\text{Rep}\mathfrak{A}$ and $\mathcal{S}(\mathfrak{A})$ denote the class of all $*$ -representations and the set of all states of \mathfrak{A} , respectively. By using the set $\{\varphi_{\mathbf{a}, \mathbf{b}} : \mathbf{a}, \mathbf{b} \in \mathbf{N}_{\geq 2}^\infty\}$ in (1.4), define the operation \otimes_φ from $\text{Rep}\mathcal{A}(\mathbf{a}) \times \text{Rep}\mathcal{A}(\mathbf{b})$ to $\text{Rep}\mathcal{A}(\mathbf{a} \cdot \mathbf{b})$ by

$$\pi_1 \otimes_\varphi \pi_2 \equiv (\pi_1 \otimes \pi_2) \circ \varphi_{\mathbf{a}, \mathbf{b}} \quad (3.1)$$

for $(\pi_1, \pi_2) \in \text{Rep}\mathcal{A}(\mathbf{a}) \times \text{Rep}\mathcal{A}(\mathbf{b})$. We see that if π_i and π'_i are unitarily equivalent for $i = 1, 2$, then $\pi_1 \otimes_\varphi \pi_2$ and $\pi'_1 \otimes_\varphi \pi'_2$ are also unitarily equivalent. Furthermore, define the operation \otimes_φ from $\mathcal{S}(\mathcal{A}(\mathbf{a})) \times \mathcal{S}(\mathcal{A}(\mathbf{b}))$ to $\mathcal{S}(\mathcal{A}(\mathbf{a} \cdot \mathbf{b}))$ by

$$\rho_1 \otimes_\varphi \rho_2 \equiv (\rho_1 \otimes \rho_2) \circ \varphi_{\mathbf{a}, \mathbf{b}} \quad (3.2)$$

for $(\rho_1, \rho_2) \in \mathcal{S}(\mathcal{A}(\mathbf{a})) \times \mathcal{S}(\mathcal{A}(\mathbf{b}))$. From (2.10), we see that

$$(\pi_1 \otimes_\varphi \pi_2) \otimes_\varphi \pi_3 = \pi_1 \otimes_\varphi (\pi_2 \otimes_\varphi \pi_3), \quad (\rho_1 \otimes_\varphi \rho_2) \otimes_\varphi \rho_3 = \rho_1 \otimes_\varphi (\rho_2 \otimes_\varphi \rho_3) \quad (3.3)$$

for each $(\pi_1, \pi_2, \pi_3) \in \text{Rep}\mathcal{A}(\mathbf{a}) \times \text{Rep}\mathcal{A}(\mathbf{b}) \times \text{Rep}\mathcal{A}(\mathbf{c})$ and for $(\rho_1, \rho_2, \rho_3) \in \mathcal{S}(\mathcal{A}(\mathbf{a})) \times \mathcal{S}(\mathcal{A}(\mathbf{b})) \times \mathcal{S}(\mathcal{A}(\mathbf{c}))$ and $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{N}_{\geq 2}^\infty$.

The following fact is a paraphrase of well-known results of tensor products of factors.

Fact 3.1 *Let π_1 and π_2 be $*$ -representations of $\mathcal{A}(\mathbf{a})$ and $\mathcal{A}(\mathbf{b})$, respectively. Then the following holds:*

- (i) *If both π_1 and π_2 are factor representations, then so is $\pi_1 \otimes_\varphi \pi_2$.*
- (ii) *The type of $\pi_1 \otimes_\varphi \pi_2$ coincides with that of $\pi_1 \otimes \pi_2$ where the type of a representation π of a C^* -algebra \mathfrak{A} means the type of the von Neumann algebra $\pi(\mathfrak{A})''$ ([6], Theorem III.2.5.27).*
- (iii) *If both π_1 and π_2 are irreducible, then so is $\pi_1 \otimes_\varphi \pi_2$.*

Proof. By the definition of \otimes_φ ,

$$(\pi_1 \otimes_\varphi \pi_2)(\mathcal{A}(\mathbf{a} \cdot \mathbf{b})) = (\pi_1 \otimes \pi_2)(\mathcal{A}(\mathbf{a}) \otimes \mathcal{A}(\mathbf{b})). \quad (3.4)$$

- (i) Since $\pi_1 \otimes \pi_2$ is also a factor representation, the statement holds from (3.4).
- (ii) By definition, the type of $\pi_1 \otimes_\varphi \pi_2$ is that of $\{(\pi_1 \otimes_\varphi \pi_2)(\mathcal{A}(\mathbf{a} \cdot \mathbf{b}))\}''$. From this and (3.4), the statement holds.
- (iii) By assumption, $\pi_1 \otimes \pi_2$ is also irreducible. From this and (3.4), the statement holds. ■

By definition, the essential part of the tensor product \otimes_φ is given by the set $\{\varphi_{\mathbf{a}, \mathbf{b}}\}$ of isomorphisms in (1.3). This type of tensor product is known yet in neither operator algebras nor the purely algebraic theory of quantum groups [16].

Remark 3.2 (i) Our terminology “tensor product of representations” is different from usual sense [12]. Remark that, for $\pi, \pi' \in \text{Rep}\mathcal{A}(\mathbf{a})$, $\pi \otimes_{\varphi} \pi' \notin \text{Rep}\mathcal{A}(\mathbf{a})$ but $\pi \otimes_{\varphi} \pi' \in \text{Rep}\mathcal{A}(\mathbf{a} \cdot \mathbf{a})$ because $\mathbf{a} \cdot \mathbf{a} \neq \mathbf{a}$ for any $\mathbf{a} \in \mathbf{N}_{\geq 2}^{\infty}$.

(ii) From Fact 3.1(iii), there is no nontrivial branch of the irreducible decomposition of the tensor product of any two irreducibles. In general, such a tensor product of the other algebra is decomposed into more than one irreducible component. For example, see Theorem 1.6 of [17].

3.2 GNS representations by product states and their tensor product formulas

We recall well-known GNS representations by product states of UHF algebras [3, 4, 28]. Let M_n be as in § 1.3 and let $M_{n,+1}$ denote the set of all positive elements in M_n whose traces are 1. Then a linear functional ω on M_n is a state of M_n if and only if ω is equal to the state ω_T which is defined as $\omega_T(x) \equiv \text{tr}(Tx)$ ($x \in M_n$) for some $T \in M_{n,+1}$ where tr denotes the trace of M_n . For $\mathbf{a} = (a_1, a_2, \dots) \in \mathbf{N}_{\geq 2}^{\infty}$, let $\mathcal{T}(\mathbf{a}) \equiv \prod_{n \geq 1} M_{a_n, +1}$. For a sequence $\mathbf{T} = (T^{(n)})_{n \geq 1} \in \mathcal{T}(\mathbf{a})$, define the state $\omega_{\mathbf{T}}$ of $\mathcal{A}(\mathbf{a})$ by

$$\omega_{\mathbf{T}}(E_{j_1, k_1}^{(a_1)} \otimes \cdots \otimes E_{j_n, k_n}^{(a_n)}) \equiv T_{k_1, j_1}^{(1)} \cdots T_{k_n, j_n}^{(n)} \quad (3.5)$$

for each $j_1, \dots, j_n, k_1, \dots, k_n$ and $n \geq 1$ where $T_{j,k}^{(n)}$'s denote matrix elements of the matrix $T^{(n)}$. Then $\omega_{\mathbf{T}}$ coincides with the product state $\bigotimes_{n \geq 1} \omega_{T^{(n)}}$.

Theorem 3.3 ([15], Remark 11.4.16) *For each $\mathbf{T} \in \mathcal{T}(\mathbf{a})$, the state $\omega_{\mathbf{T}}$ in (3.5) is a factor state, that is, if $(\mathcal{H}_{\mathbf{T}}, \pi_{\mathbf{T}}, \Omega_{\mathbf{T}})$ is the GNS triplet of $\mathcal{A}(\mathbf{a})$ by $\omega_{\mathbf{T}}$, then $\pi_{\mathbf{T}}(\mathcal{A}(\mathbf{a}))''$ is a factor.*

The factor $\mathcal{M}_{\mathbf{T}} \equiv \pi_{\mathbf{T}}(\mathcal{A}(\mathbf{a}))''$ is called an *Araki-Woods factor (or infinite tensor product of finite dimensional type I (=ITPFI) factor)* [3, 4]. Properties of $\mathcal{M}_{\mathbf{T}}$ and $(\mathcal{H}_{\mathbf{T}}, \pi_{\mathbf{T}}, \Omega_{\mathbf{T}})$ are closely studied in [3, 4, 31] and [5], respectively.

Next, we show tensor product formulas of $\pi_{\mathbf{T}}$'s in Theorem 3.3 as follows.

Theorem 3.4 *Let $\mathbf{a}, \mathbf{b} \in \mathbf{N}_{\geq 2}^{\infty}$ and let $\omega_{\mathbf{T}}$ be as in (3.5) with the GNS representation $\pi_{\mathbf{T}}$.*

(i) *For each $\mathbf{T} \in \mathcal{T}(\mathbf{a})$ and $\mathbf{R} \in \mathcal{T}(\mathbf{b})$,*

$$\omega_{\mathbf{T}} \otimes_{\varphi} \omega_{\mathbf{R}} = \omega_{\mathbf{T} \boxtimes \mathbf{R}} \quad (3.6)$$

where $\mathbf{T} \boxtimes \mathbf{R} \in \mathcal{T}(\mathbf{a} \cdot \mathbf{b})$ is defined as

$$\mathbf{T} \boxtimes \mathbf{R} \equiv (T^{(1)} \boxtimes R^{(1)}, T^{(2)} \boxtimes R^{(2)}, T^{(3)} \boxtimes R^{(3)}, \dots) \quad (3.7)$$

for $\mathbf{T} = (T^{(n)})$ and $\mathbf{R} = (R^{(n)})$.

- (ii) For each $\mathbf{T} \in \mathcal{T}(\mathbf{a})$ and $\mathbf{R} \in \mathcal{T}(\mathbf{b})$, $\pi_{\mathbf{T}} \otimes_{\varphi} \pi_{\mathbf{R}}$ is unitarily equivalent to $\pi_{\mathbf{T} \boxtimes \mathbf{R}}$.

Proof. (i) By definition, the statement holds from direct computation.

(ii) Let $\mathbf{a} \in \mathbf{N}_{\geq 2}^{\infty}$. For $\mathbf{T} \in \mathcal{T}(\mathbf{a})$, let $(\mathcal{H}_{\mathbf{T}}, \pi_{\mathbf{T}}, \Omega_{\mathbf{T}})$ denote the GNS triplet by the state $\omega_{\mathbf{T}}$. Define the GNS map $\Lambda_{\mathbf{T}}$ [26, 27] from $\mathcal{A}(\mathbf{a})$ to $\mathcal{H}_{\mathbf{T}}$ by $\Lambda_{\mathbf{T}}(x) \equiv \pi_{\mathbf{T}}(x)\Omega_{\mathbf{T}}$ for $x \in \mathcal{A}(\mathbf{T})$. Let $\mathbf{a}, \mathbf{b} \in \mathbf{N}_{\geq 2}^{\infty}$. For $\mathbf{T} \in \mathcal{T}(\mathbf{a})$ and $\mathbf{R} \in \mathcal{T}(\mathbf{b})$, define the unitary $U^{(\mathbf{T}, \mathbf{R})}$ from $\mathcal{H}_{\mathbf{T} \boxtimes \mathbf{R}}$ to $\mathcal{H}_{\mathbf{T}} \otimes \mathcal{H}_{\mathbf{R}}$ by

$$U^{(\mathbf{T}, \mathbf{R})} \Lambda_{\mathbf{T} \boxtimes \mathbf{R}}(x) \equiv (\Lambda_{\mathbf{T}} \otimes \Lambda_{\mathbf{R}})(\varphi_{\mathbf{a}, \mathbf{b}}(x)) \quad (x \in \mathcal{A}(\mathbf{a} \cdot \mathbf{b})). \quad (3.8)$$

Since $\varphi_{\mathbf{a}, \mathbf{b}}$ is bijective, $U^{(\mathbf{T}, \mathbf{R})}$ is well-defined as a unitary, and we see that

$$U^{(\mathbf{T}, \mathbf{R})} \pi_{\mathbf{T} \boxtimes \mathbf{R}}(x) (U^{(\mathbf{T}, \mathbf{R})})^* = (\pi_{\mathbf{T}} \otimes_{\varphi} \pi_{\mathbf{R}})(x) \quad (x \in \mathcal{A}(\mathbf{a} \cdot \mathbf{b})). \quad (3.9)$$

Hence two representations $\pi_{\mathbf{T} \boxtimes \mathbf{R}}$ and $\pi_{\mathbf{T}} \otimes_{\varphi} \pi_{\mathbf{R}}$ are unitarily equivalent. \blacksquare

From Theorem 3.4, the tensor product \otimes_{φ} is compatible with product states and their GNS representations. More precisely, for the following two semigroups (\mathcal{T}, \boxtimes) and $(\mathcal{S}, \otimes_{\varphi})$, the map

$$\mathcal{T} \equiv \bigcup_{\mathbf{a} \in \mathbf{N}_{\geq 2}^{\infty}} \mathcal{T}(\mathbf{a}) \ni \mathbf{T} \mapsto \omega_{\mathbf{T}} \in \mathcal{S} \equiv \bigcup_{\mathbf{a} \in \mathbf{N}_{\geq 2}^{\infty}} \mathcal{S}(\mathcal{A}(\mathbf{a})) \quad (3.10)$$

is a semigroup homomorphism. Let $R_{\mathbf{a}}$ denote the set of all unitary equivalence classes in $\text{Rep} \mathcal{A}(\mathbf{a})$. Then

$$\mathcal{T} \ni \mathbf{T} \mapsto [\pi_{\mathbf{T}}] \in R \equiv \bigcup_{\mathbf{a} \in \mathbf{N}_{\geq 2}^{\infty}} R_{\mathbf{a}} \quad (3.11)$$

is also a semigroup homomorphism from (\mathcal{T}, \boxtimes) to (R, \otimes_{φ}) where $[\pi]$ denotes the unitary equivalence class of a representation π .

3.3 Examples

In this subsection, we show examples of Theorem 3.4 for special UHF algebras. Let $\mathcal{A}(\mathbf{a})$ and $\mathcal{T}(\mathbf{a})$ be as in § 3.2. For $n \geq 1$, let

$$\mathbf{n} \equiv (n, n, n, \dots) \in \mathbf{N}^\infty \quad (3.12)$$

and let

$$UHF_n \equiv \mathcal{A}(\mathbf{n}) = (M_n)^{\otimes \infty} \quad (n \geq 2). \quad (3.13)$$

Then UHF_n is the UHF algebra of Glimm's type $\{n^l\}_{l \geq 1}$.

Let $\{E_{i,j}^{(n)}\}$ be as in § 1.3. For $j \in \{1, \dots, n\}$, define $F_j^{(n)} \equiv E_{j,j}^{(n)}$. Then $F_j^{(n)} \in M_{n,+1}$ for each j . For $J = (j_1, j_2, \dots) \in \{1, \dots, n\}^\infty$, define $\mathbf{T}(J) \equiv (F_{j_1}^{(n)}, F_{j_2}^{(n)}, \dots) \in \mathcal{T}(\mathbf{n})$. From (3.5), we see that

$$\omega_{\mathbf{T}(J)}(E_{l_1,k_1}^{(n)} \otimes \dots \otimes E_{l_m,k_m}^{(n)}) \equiv \delta_{l_1,j_1} \dots \delta_{l_m,j_m} \delta_{k_1,j_1} \dots \delta_{k_m,j_m} \quad (3.14)$$

for each $l_1, \dots, l_m, k_1, \dots, k_m \in \{1, \dots, n\}$ and $m \geq 1$. For $J = (j_n)_{n \in \mathbf{N}}$, $J' = (j'_n)_{n \in \mathbf{N}} \in \{1, \dots, n\}^\infty$, we write $J \approx J'$ if there exists an integer $n_0 \geq 1$ such that $j_r = j'_r$ for each $r \geq n_0$.

Proposition 3.5 *For $J \in \{1, \dots, n\}^\infty$, let $\pi_{\mathbf{T}(J)}$ denote the GNS representation of UHF_n by the state $\omega_{\mathbf{T}(J)}$ in (3.14), and let $P_n[J]$ denote the unitary equivalence class of $\pi_{\mathbf{T}(J)}$. Then the following holds:*

- (i) *For each $J \in \{1, \dots, n\}^\infty$, $\pi_{\mathbf{T}(J)}$ is irreducible.*
- (ii) *For $J, J' \in \{1, \dots, n\}^\infty$, $P_n[J] = P_n[J']$ if and only if $J \approx J'$.*
- (iii) *For each $J = (j_l) \in \{1, \dots, n\}^\infty$, $K = (k_l) \in \{1, \dots, m\}^\infty$ and $n, m \geq 2$,*

$$P_n[J] \otimes_\varphi P_m[K] = P_{nm}[J \star K] \quad (3.15)$$

where $J \star K \in \mathcal{T}(\mathbf{n} \cdot \mathbf{m})$ is defined by

$$J \star K \equiv (m(j_1 - 1) + k_1, m(j_2 - 1) + k_2, m(j_3 - 1) + k_3, \dots). \quad (3.16)$$

Proof. (i) Since the state $\omega_{\mathbf{T}(J)}$ is the product state of pure states, $\omega_{\mathbf{T}(J)}$ is also pure. Hence the statement holds.

(ii) From (2.1) in [2] (see also [7]), the statement holds.

(iii) Recall \boxtimes in (3.7). Then we can verify that $\mathbf{T}(J) \boxtimes \mathbf{T}(K) = \mathbf{T}(J \star K)$ for each $J \in \{1, \dots, n\}^\infty$ and $K \in \{1, \dots, m\}^\infty$. From this and Theorem 3.4(ii), the statement holds. ■

In Theorem 1.6 of [17], “ $J \star K$ ” is written as the different notation “ $J \cdot K$.”

Example 3.6 From Proposition 3.5, $\mathbf{P} \equiv \bigcup_{n \geq 2} \{P_n[J] : J \in \{1, \dots, n\}^\infty\}$ is a semigroup of unitary equivalence classes of irreducible representations with respect to the product \otimes_φ . For $n \geq 1$, let \mathbf{n} be as in (3.12). Then

$$P_2[\mathbf{1}] \otimes_\varphi P_2[\mathbf{2}] = P_4[\mathbf{2}], \quad P_2[\mathbf{2}] \otimes_\varphi P_2[\mathbf{1}] = P_4[\mathbf{3}] \quad (3.17)$$

because $\mathbf{1} \star \mathbf{2} = \mathbf{2}$ and $\mathbf{2} \star \mathbf{1} = \mathbf{3}$. Since $\mathbf{2} \not\approx \mathbf{3}$, $P_4[\mathbf{2}] \neq P_4[\mathbf{3}]$. Hence $(\mathbf{P}, \otimes_\varphi)$ is non-commutative.

The class $P_n[J]$ in Proposition 3.5(ii) coincides with the restriction of a permutative representation of the Cuntz algebra \mathcal{O}_n on the UHF subalgebra of $U(1)$ -gauge invariant elements in \mathcal{O}_n , which is called an atom [2, 8]. This class contains only type I representations of UHF_n . Relations with representations of Cuntz algebras and quantum field theory are well studied [2, 8].

From Example 3.6 and pure states associated with $P_2[\mathbf{1}]$ and $P_2[\mathbf{2}]$, the following holds.

Corollary 3.7 *Let $\mathcal{S}(\mathcal{A}(\mathbf{a}))$ and $\mathbf{R}_\mathbf{a}$ be as in § 3.1.*

- (i) *There exist $\mathbf{a}, \mathbf{b} \in \mathbf{N}_{\geq 2}^\infty$, and states $\omega \in \mathcal{S}(\mathcal{A}(\mathbf{a}))$ and $\omega' \in \mathcal{S}(\mathcal{A}(\mathbf{b}))$ such that $\omega \otimes_\varphi \omega' \neq \omega' \otimes_\varphi \omega$.*
- (ii) *There exist $\mathbf{a}, \mathbf{b} \in \mathbf{N}_{\geq 2}^\infty$, and classes $[\pi] \in \mathbf{R}_\mathbf{a}$ and $[\pi'] \in \mathbf{R}_\mathbf{b}$ such that $[\pi] \otimes_\varphi [\pi'] \neq [\pi'] \otimes_\varphi [\pi]$.*

From Corollary 3.7(i) and (ii), we say that \otimes_φ is non-symmetric. In other words, two semigroups $(\mathcal{S}, \otimes_\varphi)$ and $(\mathbf{R}, \otimes_\varphi)$ are non-commutative. These non-commutativities come from the non-commutativity of the Kronecker product of matrices in Remark 1.2.

Problem 3.8 (i) Generalize the tensor product in (3.1) to that of representations of approximately finite dimensional (=AF) algebras which are not always UHF algebras.

- (ii) Reconstruct UHF algebras from $(\mathcal{S}, \otimes_\varphi)$ and $(\mathbf{R}, \otimes_\varphi)$, and show a Tatsuuma duality type theorem for UHF algebras [33].

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